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UNSYMMETRICAL BENDING OF BEAMS WITH AND WITHOUT LATERAL BRACING

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UNSYMMETRICAL BENDING OF BEAMS WITH AND WITHOUT LATERAL BRACING

Lev Zetlin,[1] A.M. ASCE, and George Winter,[2] M. ASCE

SYNOPSIS

Beams loaded obliquely with respect to the principal axes deflect perpendicular to, as well as in the plane of the load; in addition, the beam twists about a longitudinal axis. This situation occurs if symmetrical sections (such as I-beams) are loaded in a plane other than a principal one, or if unsymmetrical sections are loaded in this manner (such as Z-beams loaded in the plane of the web). If rotations are held within close limits, their influence can be neglected. For this frequent case computation of deflections and stresses by ordinary methods is lengthy since the neutral axis is neither a principal axis nor perpendicular to the plane of loading. In this paper simple equations are derived which are identical in form with the elementary formulas for symmetrical bending and which permit the calculation of deflections and stresses with great ease. The method is then used to analyze the case of an obliquely loaded beam braced at intermediate points against lateral deflection. It is problems of this kind which are difficult to handle by existing methods and for which the proposed approach furnishes extremely simple solutions. A sample computation is given of a Z-beam so braced, and loaded in the plane of the web; its results are compared with test data, and close agreement is established.

INTRODUCTION

This paper presents a simple method of analyzing flexural members subjected to loads oblique to the principal planes of the section. In such cases it is known that the member will bend in the plane of the loads as well as out of this plane. It is less well known that such oblique bending always involves simultaneous rotation.[3]

The usual methods for designing such members, even when the rotations are neglected, are cumbersome inasmuch as most of them require the determination of the neutral axis which is inclined to the plane of the loads, or the separate calculation of bending in the two principal planes. The amount of work is increased if deflections have to be computed. Further mathematical complications are involved if such members are continuous over more than two supports. Finally, a common practical case is that where some of the

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3. Strength of Slender Beams, by George Winter, Trans. ASCE, vol. 109, p. 1323, 1944.

supports provide reactions in two perpendicular directions (e.g., vertical and horizontal reactions), whereas the remaining supports provide reactions in one direction only (e.g., horizontal bracing). An example of such a member is shown in Fig. 4a. In this figure, a Z-beam is loaded in the plane of its web, is supported vertically at two points "a" and "b", and is braced horizontally at three points "a", "b" and "c".

This paper is concerned only with such cases of unsymmetrical bending where secondary effects due to rotations may be neglected.

In section I, formulas are presented for deflection computations of simply supported unsymmetrically bent members. These formulas have the same general form as the elementary formulas for bending in the principal planes and hence can be applied by a designer without detained knowledge of unsymmetrical bending.

In Section II, intermediately braced flexural members are discussed. A method is indicated for computing deflections and stresses in these statically indeterminate cases. This method is analogous to conventional methods applied to statically indeterminate flexural members bent in their principal plane.

The presented formulas and methods apply to unsymmetrical bending of any cross-sectional shape. For illustration, however, a Z-section is employed throughout the paper.

I. Simply Supported Z-Beam

A Z-beam which is loaded in the plane of the web (assumed to be vertical) is shown in Fig. 1a and its cross-section in Fig. 1b. The positive directions of the x, y and z axes are as indicated. A load is positive if it acts in the positive direction of x and y. The bending moment M is positive if it produces curvature convex towards the positive direction of x and y. Tensile stresses are positive; compressive stresses, negative. Ends "a" and "b" of the beam are simply supported, i.e., they can rotate about the x and the y axes but not about the z-axis. This is the usual method of support for such members.

Under external loads in the y - z plane the Z-beam will bend both vertically and horizontally, and will twist about the z-axis. The latter effect, with the additional stresses it produces, is usually of secondary nature. It will be discussed briefly in Section II. For the Z-beam in Fig. 1, only the stresses and the deflections due to bending in the vertical and horizontal planes, which are of primary nature, will be discussed. At this stage it will be mentioned that the twist is ordinarily a secondary effect if the shear center of the section coincides with its centroid, if the plane of the external loads contains the longitudinal axis through the centroid, and if the deflections perpendicular to the plane of the loads are small.

In unbraced beams, even if the first two conditions are satisfied, lateral deflections and correspondingly high secondary stresses may occur for relatively short spans. It is this very fact which necessitates bracing in such cases. While the method, below, is not entirely reliable for such unbraced beams, since it results in computed maximum stresses which are smaller than actual stresses, it is sufficiently accurate to allow the designer to determine whether lateral bracing is needed in a given case of this nature.

The designer is usually interested in the stresses and in the deflections. Instead of computing these quantities by determining the inclination of the neutral axis or the load components parallel to the principal axes, as is ordinarily done, a relatively straightforward procedure will be presented. It

permits one to calculate the deflections and the stresses by direct application of the usual elementary formulas for symmetrical bending.

The method consists in the following: The member is considered as being loaded by the actual loads and by "fictitious loads" in a centroidal plane perpendicular to that of the actual loads. Stresses caused by, and deflections in the directions of, these two types of loads are computed by the elementary formulas for symmetrical bending. However, "modified moments of inertia" I_{mx} and I_{my} are used instead of the usual quantities I_x and I_y in connection with bending about the x- and y-axis, respectively. The actual stresses and deflections are the sums of the corresponding values computed in this manner for the actual and for the fictitious loads. The "fictitious loads" and "modified moments of inertia" employed in this method are defined below. The theory justifying their use is given in the Appendix.

a) Fictitious Loads:

For an actual load in the y-z plane, assume a fictitious load in the x-z plane at the same location along the span as the actual load and of intensity $(-I_{xy}/I_x)$ times the actual load, as is shown in Figs. 2a and 2c.

Similarly, if the actual load is in the x-z plane, assume fictitious load in the y-z plane at the same location along the span as the actual load and of intensity of $(-I_{xy}/I_y)$ times the actual load, as is shown in Figs. 2b and 2d.

Note that with the orientation of the section as in Figs. 2a and 2b, I_{xy} is a positive quantity. Hence with the actual loads in the y-z plane towards +y, the fictitious load acts towards -x; with the actual loads in the x-z plane towards +x, the fictitious load acts towards -y. On the other hand, with the orientation of the section as in Figs. 2c and 2d, I_{xy} is a negative quantity. Hence with the actual loads in the y-z plane towards +y, the fictitious load acts towards +x; with the actual loads in the x-z plane towards -x, the fictitious load acts towards -y.

b) Modified Moments of Inertia:

The "modified moments of inertia" about the x- and the y-axes, respectively, are defined as follows:

About the x-axis:

$$I_{mx} = \frac{I_x I_y - I_{xy}^2}{I_y} \qquad \text{Eqn. (1a)}$$

About the y-axis:

$$I_{my} = \frac{I_x I_y - I_{xy}^2}{I_x} \qquad \text{Eqn. (1b)}$$

where I_x and I_y are the usual moments of inertia about the x- and the y-axis respectively, and I_{xy} is the product of inertia about these axes. Derivation of the relationships in Eqns. (1a) and (1b) is indicated in the Appendix.

The application of the above procedure to the calculation of the vertical and horizontal deflections and of stresses at midspan of the Z-beam in Fig. 1 will be illustrated.

In this case, a concentrated load P is applied at midspan in the plane of the web, i.e. in the y-z plane. Hence, the fictitious load $(-\frac{I_{xy}}{I_x} P)$ will also be

applied at the center of the span but in the x-z plane and will act towards -x. Thus the beam will undergo simultaneously bending in the y-z and the x-z planes, in each case by a concentrated center load. In each plane the bending of the beam will be treated independently.

Consider first bending in the y-z plane.

The vertical deflection at the center of the span is:

$$\delta_y = \frac{PL^3}{48\ EI_{mx}} \qquad \text{Eqn. (2a)}$$

The bending moment at the center is obviously $M_x = PL/4$ hence the stress at point "A" is:

$$\sigma_1 = \frac{M_x\,(-h/2)}{I_{mx}} = \frac{(PL/4)(\,-h/2)}{I_{mx}} \qquad \text{Eqn. (2b)}$$

Next consider bending in the x-z plane.

The horizontal deflection at the center of the span is:

$$\delta_x = \frac{(-\frac{I_{xy}}{I_x}P)\ L^3}{48\ E\ I_{my}} \qquad \text{Eqn. (2c)}$$

The (fictitious) bending moment at the center is $M_y = (-\frac{I_{xy}}{I_x}P)\ L/4$; hence the stress at point "A" is:

$$\sigma_2 = \frac{M_y\,(-b)}{I_{my}} = \frac{(-\frac{I_{xy}}{I_x}P)L/4(-b)}{I_{my}} \qquad \text{Eqn. (2d)}$$

The total, oblique deflection is the vector sum of δ_x and δ_y.

The total stress at point "A" will be the algebraic sum of (2b) and (2d), i.e.:

$$\sigma_{total} = \sigma_1 + \sigma_2 = \frac{(PL/4)(-h/2)}{I_{mx}} + \frac{(-\frac{I_{xy}}{I_x}P)L/4(-b)}{I_{my}} \qquad \text{Eqn. (2e)}$$

For further illustration, the center deflections in the x-z and the y-z planes of the Z-beam in Fig. 3 will be determined. It is similar to that in Fig. 1, but loaded perpendicular to the plane of the web. In this case, the actual load is a concentrated load-Q at midspan in the x-z plane. The fictitious load is a concentrated load $(\frac{I_{xy}}{I_y}Q)$ at midspan in the y-z plane towards +y as shown in Fig. 3. Therefore,

Center deflection in the x-z plane:

$$\delta_x = \frac{(-Q)L^3}{48\ EI_{my}} \qquad \text{Eqn. (2f)}$$

Center deflection in the y-z plane:

$$\delta_y = \frac{(\frac{I_{xy}}{I_y}\ Q)L^3}{48\ EI_{mx}} \qquad \text{Eqn. (2g)}$$

The stresses at any point in a cross-section may be computed in a way similar to Eqns. (2b), (2d) and (2e), with due consideration to the signs. The total stress at point "A" is:

$$\delta_{total} = \frac{(-QL/4)(-b)}{I_{my}} + \frac{(\frac{I_{xy}}{I_y}\ Q)L/4(-h/2)}{I_{mx}} \qquad \text{Eqn. (2h)}$$

II. Simply Supported Beams Braced Laterally at Intermediate Points

In dealing with unsymmetrical bending in Section I, the secondary stresses were disregarded. If the distance between the supports is large and the torsional rigidity low, the secondary stresses can attain considerable magnitude and may no longer be neglected; moreover, the rotations as well as the lateral deflections may reach values which can not be permitted in practical application. Evaluation of the secondary stresses, rotations etc. which take place in unsymmetric bending have been treated in several previous papers.[3,4,5] It is laborious and frequently prohibitive in routine design. More important, if secondary stresses, rotations and deflections are not limited to small amounts, they sharply reduce the useful capacity of the member. Therefore, it is desirable to provide intermediate braces along the span in such a way that the rotations, and the secondary effects due to them, are limited to practically negligible magnitudes. For example, if the Z-beam in Fig. 4 is braced horizontally at an intermediate point "c", the rotations of the beam are decreased, which in turn decrease the secondary effects. A similar approach has been developed for the related case of channel-beams loaded in the plane of the web.[6]

The question of required number and location of these intermediate bracings, is beyond the scope of this paper. This problem was treated by the

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4. Thin-Walled Members in Combined Torsion and Flexure, by W. Lansing, ASCE Trans., vol. 118, p. 128, 1953 (Also, Ph.D. Thesis, 1949, Cornell Univ., by same author).
5. Contributions to the Theory of Combined Flexure and Torsion, by Robert B. McCalley, Ph.D. Thesis, Cornell Univ., Ithaca, N.Y., 1952.
6. Performance of Laterally Loaded Channel Beams, by G. Winter, W. Lansing, and R. B. McCalley, Cornell Univ. Eng'g. Exp. Station, Reprint No. 33, 1950. First published in Research, Engineering Structures Supplement, Colston Papers, Vol. II, p. 179, London, 1949.

writers in reference (7) in reports on a research project carried out at Cornell University for the American Iron and Steel Institute.[7]

The present section of the paper will be concerned with, 1) a discussion of the behaviour of a braced Z-beam with major emphasis on the physical action of an intermediate brace, and 2) a numerical analysis of deflections and stresses in an intermediately braced Z-beam. In each case the secondary moments and stresses are neglected and hence the principle of superposition applies to the outlined procedure (see Appendix). Although a Z-section is used as an example, the same approach may be used for any other sectional shape of a beam in oblique bending.

Physical Action of an Intermediate Brace

An intermediately braced Z-beam loaded in the plane of the web, is shown in Fig. 4a.

Owing to the brace, cross-section "c" of the beam at the location of the brace is prevented from rotating about the z-axis and from displacing laterally in the x-direction. It is not prevented, however, from rotating either about the x- or the y-axis, or from displacing in the y-direction.

With the application of the external loads, the brace will exert a horizontal reaction on the beam and thus the beam will be subjected to a bending moment M_y about the y-axis, in addition to the bending moment M_x about the x-axis. The total bending stresses at a point in any cross-section of the beam will therefore be composed of those due to M_x and M_y. These stresses can be determined only if the horizontal reaction from the brace at "c" is known. Similarly, both the vertical and the horizontal deflections anywhere along the beam will be influenced by the horizontal reaction from the brace whose magnitude must be known in order to be able to compute any of these deflections. The magnitude of the horizontal reaction, denoted by F_c in Fig. 4a, can be deduced from the following:

If the brace is removed, the externally applied vertical load will produce a certain horizontal deflection at the brace location, i.e. at point "c", Fig. (4a). However, due to the presence of the brace which exerts a horizontal reaction F_c, the actual horizontal deflection at "c" is zero. In other words, the magnitude of the force F_c is such that the horizontal deflection at "c" caused by F_c is equal and opposite to the horizontal deflection caused by the vertical load w.

In the discussion just given, only the real loads and the real bending moments have been considered. However, the behaviour of the beam in Fig. (4a) can also be described in terms of the fictitious loads and the fictitious bending moments caused by these loads. The final result in each case will be identical. But the numerical computations of deflections and stresses are much simpler if fictitious loads are used. This will be illustrated by a numerical example which will follow later.

The method of fictitious loads applied to the same braced Z-beam of Fig. (4a) is this: There are real loads w in the y-z plane and F_c in the x-z plane. In accordance with the explanation given in section I of this paper, there will arise corresponding fictitious loads $F_c\ (-I_{xy}/I_y)$ in the y-z plane and $w\ (-I_{xy}/I_x)$ per unit length in the x-z plane as is shown, respectively, in Figs. (4b) and (4c). Bending in each plane may now be treated independently; in each plane the beam is loaded simultaneously by the real and the fictitious loads.

7. L. Zetlin and G. Winter, 64th and 67th Progress Reports to the American Iron and Steel Institute on Tests on Light Gage Beams in Cold Formed Steel, Cornell Univ., Ithaca, N. Y., April 1952, and August 1954.

In the y-z plane, the beam is simply supported and loaded by the real distributed load w per unit length and by the fictitious concentrated load F_c $(-I_{xy}/I_y)$. Deflections in this plane, and stresses caused by these loads can be found by elementary formulas for symmetrical bending using I_{mx} as the moment of inertia.

In the x-z plane, the beam rests on three unyielding supports and is loaded by the fictitious load w $(-I_{xy}/I_x)$ per unit length. The reaction (i.e. force F_c), as well as the stresses and the deflections in the x-z plane can be found by standard formulas or methods for continuous beams, by using I_{my} for the moment of inertia.

Finally, total stresses and deflections are the sums of individual stresses and deflections caused by the loads, real and fictitious, in the y-z and the x-z planes.

Numerical Example and Test Results

In this example numerical computations for deflections and stresses will be made for the braced Z-beam shown in Fig. (5a). This case represents one of the several tests carried out at Cornell University in connection with the work referred to in (7). Test values for comparison with computations will be given.

The required properties of the Z-section and other information relating to the beam are as follows:

$h/2 = 3.713$ inches
$I_x = 9.07$ in.^{4}
$I_y = 1.41$ in.^{4}
$I_{xy} = 2.60$ in.^{4}
$I_{mx} = 4.28$ in.^{4}
$I_{my} = 0.68$ in.^{4}
Design Load, P_d = 522 lbs.
Clear Span, L = 210 in.
Yield Stress = 42400 psi.

The following investigation will be carried out:

(a) Horizontal and vertical deflections at midspan under the design load.
(b) Total stress at point "B" at the design load (the compressive stress at this point is largest in the beam).
(c) Determination of Yield Load, P_y; i.e., the load at which the maximum stress in the beam is equal to the yield stress.

Solution:

(a) Consider first bending in the x-z plane, Fig. (5b). In this plane the beam rests on four unyielding supports and is loaded by the fictitious loads $P_d\,(I_{xy}/I_x) = 150$ lbs. The brace reactions F_c can be computed by standard formulas for continuous beams. Such a computation results in,

$$F_c = 166 \text{ lbs.}$$

$$F_a = F_b = -16 \text{ lbs. (in the same direction as the fictitious load)} \qquad \text{Eqn. (3)}$$

The horizontal deflection at midspan is computed by the usual methods for deflections in continuous beams, using I_{my} as the moment of inertia. In this case, since the forces F_c are known, the deflection can be easily computed for a simply supported beam of 210 inches span and loaded as in Fig. (5b). Namely,

Deflection towards -x due to the two fictitious loads of 150 lbs. each:[8]

$$\delta_{x_1} = \frac{150 \times 92}{24 \times EI_{my}} (3 \times 210^2 - 4 \times 92^2) = 2.88 \text{ inches} \quad \textbf{Eqn. (4a)}$$

Deflection towards +x due to the brace reaction F_c = 166 lbs. each:[8]

$$\delta_{x_2} = \frac{166 \times 70}{24 \times EI_{my}} (3 \times 210^2 - 4 \times 70^2) = 2.83 \text{ inches} \quad \textbf{Eqn. (4b)}$$

Hence the total horizontal deflection at midspan (towards -x) is given by the algebraic sum of Eqns. (4a) and (4b):

$$\delta_{x_{total}} = \delta_{x_1} + \delta_{x_2} = 2.88 - 2.83 = 0.05 \text{ in.} \quad \textbf{Eqn. (4c)}$$

In the test, the measured horizontal deflection at this load was 0.08 inches.

To find the vertical deflection, consider bending in the y-z plane. In this plane, the beam is loaded as shown in Fig. (5c) by real loads P_d = 522 lbs., and by opposite fictitious loads F_c (I_{xy}/I_y) = 166 (2.60/1.41) = 304 lbs.

Vertical deflection at midspan caused by the real loads (towards +y):[8]

$$\delta_{y_1} = \frac{522 \times 92}{24 \times EI_{mx}} (3 \times 210^2 - 4 \times 92^2) = 1.550 \text{ inches} \quad \textbf{Eqn. (5a)}$$

Vertical deflection at midspan caused by the fictitious loads (towards -y):[8]

$$\delta_{y_2} = \frac{304 \times 70}{24 \times EI_{mx}} (3 \times 210^2 - 4 \times 70^2) = 0.793 \text{ inches} \quad \textbf{Eqn. (5b)}$$

Hence the total vertical deflection at midspan (towards +y) is given by the algebraic sum of Eqns. (5a) and (5b):

$$\delta_{y_{total}} = \delta_{y_1} + \delta_{y_2} = 1.55 - 0.793 = 0.757 \text{ inches} \quad \textbf{Eqn. (5c)}$$

In the test, the measured vertical deflection at this load was 0.760 in.

From the comparison of Eqns. (5a) and (5b) it is seen that the presence of braces reduces the vertical deflection by at least one-half the amount of that in an unbraced Z-beam.

Furthermore, from comparison of Eqns. (4a) and (4b) it is seen that the presence of lateral braces reduces the horizontal deflections from an entirely prohibitive value (2.88 in.) to a negligible amount (0.05 in.). In the unbraced beam the large lateral deflections would result in sizeable torques which would make inapplicable the proposed approximate method for determining stresses. In contrast, the presence of braces makes the torques sufficiently small to make the method reliable. Thus, the determination of deflections by the proposed method, while to some degree approximate, provides the means for judging (a) whether lateral deflections are so large as to require bracing and (b) whether torques due to lateral deflections are sufficiently small to make their influence negligible in determining stresses.

8. AISC Manual of Steel Construction, 1948, page 368, formula 9.

(b) Stress at point "B" due to bending in the x-z plane is obviously zero since the point is on the y-axis (Fig. (5a)). Hence bending in the y-z plane only has to be considered.

The bending moment at midspan caused by the real and the fictitious loads is:

$$M_x = (522 \times 92 - 304 \times 70) = 26745 \text{ lbs. in.} \qquad \text{Eqn. (6)}$$

hence the compressive stress at point "B":

$$\sigma = \frac{M_x(h/2)}{I_{mx}} = \frac{26745 \times 3.713}{4.28} = 23200 \text{ psi.} \qquad \text{Eqn. (7)}$$

It might be of interest to compare the stress in Eqn. (7) with the stress at the same point if the Z-beam under consideration were 1) constrained to bend in the y-z plane, and 2) not braced between ends at the intermediate sections. The first case is realized if vertical guides or continuous bracing are provided.

For the constrained beam, the simplest approach is to consider only the real loads and moments caused by them, and to use the usual moment of inertia I_x. The bending moment at midspan $M_x = 522 \times 92 = 48025$ lbs. in., and the stress at point "B",

$$\sigma = \frac{M_x(h/2)}{I_x} = \frac{48025 \times 3.713}{9.07} = 19720 \text{ psi.} \qquad \text{Eqn. (8)}$$

Another way to compute this stress by the method of fictitious loads is given in the Appendix.

For the unbraced beam, only the real moment of 48025 lbs. in. has to be considered and I_{mx} should be used. (The bending moment due to the fictitious load in the x-z plane will not contribute any stress at point "B", since this point is on the y-axis). Hence, the stress at point "B" would then be:

$$\sigma = \frac{M_x(h/2)}{I_{mx}} = \frac{48025 \times 3.713}{4.28} = 41700 \text{ psi.} \qquad \text{Eqn. (9)}$$

The stress in Eqn. (9) is obviously less than the actual stress. As has been shown before, in an unbraced beam of the given dimensions the lateral deflections are considerable (see Eqn. 4a) and cause appreciable rotations of the beam. Owing to these rotations, the secondary stresses are of such magnitude that they can no longer be neglected. However, comparison of Eqn. (9) with Eqn. (7) indicates that in the unbraced beam the maximum stress is of the order of at least twice that in the braced beam. This and the large lateral deflections and rotations would preclude the practical use of such an unbraced beam.

On the other hand, the braced beam in Fig. (5a) has practically negligible lateral deflections (see Eqn. (4c)) and hence the secondary effects due to twist are small and can be neglected. Hence, the value of the stress in Eqn. (7) should be very close to the actual stress that would occur in a braced beam.

Comparison of Eqns. (7) and (8) shows that this Z-beam braced only at two intermediate sections has a maximum stress 18% higher than in a corresponding continuously braced beam.

(c) Since the principle of superposition holds for the method outlined in this paper, the stresses are linear functions of the externally applied loads. Hence, the ratio of the yield load (P_y) to the design load (P_d) will be the same as the ratio of the yield stress (42400 psi) to the stress at the design load (23200 psi). Or,

$$\frac{P_y}{P_d} = \frac{42400}{23200} = 1.83, \text{ and } P_y = 522 \times 1.83 = 955 \text{ lbs.}$$

At this load the yield stress of 42400 psi. is reached only at point "B" of the cross-section. All other fibers are at a lower stress. In a ductile material failure is therefore to be expected at a load somewhat higher than that causing such first localized yielding.

The ultimate load by test was 1045 lbs. Hence, the agreement of this elementary theory and the test information is satisfactory for such a braced beam.

CONCLUSIONS

A simple, elementary method is presented for computing stresses and deflections in beams loaded obliquely to the principal planes. The method is applicable as long as stresses and deformations caused by twisting, which always accompanies oblique bending, are negligible.

The method is particularly convenient for the practical case of beams braced at one or more points between supports against deflection perpendicular to the plane of the load.

The case of a Z-beam loaded in the web and intermediately braced, probably the most frequent example of the problem under discussion, is used for a sample computation, whose results are compared with test data and show very satisfactory agreement.

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APPENDIX

Theory

In the following, a Z-section is used as an illustration; the discussion, however, may apply to any shape.

(1) Deflections: Modified Moments of Inertia and Fictitious Loads.

(a) Loads in the y-z plane:

Referring to Fig. 1, with loads in the plane of the web as shown, any cross-section will tend to displace both in the y and the x -directions. Due to the bending in the x-direction the external loads displace laterally with

respect to the vertical reactions at "a" and "b". This displacement causes twisting moments along the beam. Consequently, cross-sections of the beam between the end supports will rotate. Due to these rotations the external load will produce additional bending moments about the rotated y-axis, since the load assumed to remain vertical, now has a component in the direction of the rotated x-axis. These latter bending moments together with the previously mentioned twisting moments were defined in Section I of this paper as "secondary moments", and the stresses due to them as "secondary stresses".

In the following derivation of Eqns. (1), the secondary moments will be disregarded. This is equivalent to assuming that the Z-beam of Fig. 1 would only bend about the x and the y-axes, while the web which was vertical before the application of the loads is assumed to remain vertical after the application of the loads.

If r_x and r_y denote the radii of curvature in the y-z and x-z planes respectively, it has been shown[9] that the following relationships exist:

$$\frac{1}{r_y} = -\frac{I_{xy}}{I_y} \cdot \frac{1}{r_x} \qquad \text{(i)}$$

and

$$M_x = \left(\frac{E\,I_x}{r_x} + \frac{E\,I_{xy}}{r_y}\right) = \frac{E}{r_x}\left(\frac{I_x I_y - I_{xy}^2}{I_y}\right) = -\frac{E}{r_y}\left(\frac{I_x I_y - I_{xy}^2}{I_{xy}}\right) \qquad \text{(ii)}$$

where M_x is the bending moment about the x-axis. For loads in the y-z plane M_x is the only real moment acting on the beam.

From (ii) it follows that:

$$\frac{1}{r_x} = -\frac{d^2\delta_y}{dz^2} = \frac{M_x}{E}\left(\frac{I_y}{I_x I_y - I_{xy}^2}\right) \qquad \text{(iii)}$$

and

$$\frac{1}{r_y} = -\frac{d^2\delta_x}{dz^2} = -\frac{M_x}{E}\left(\frac{I_{xy}}{I_x I_y - I_{xy}^2}\right) \qquad \text{(iva)}$$

(iva) may be written as:

$$\frac{1}{r_y} = -\frac{d^2\delta_x}{dz^2} = \frac{M_x(-I_{xy}/I_x)}{E}\left(\frac{I_x}{I_x I_y - I_{xy}^2}\right) \qquad \text{(ivb)}$$

If one now introduces the modified moments of inertia as defined in Eqs. (1), namely,

$$I_{mx} = \frac{I_x I_y - I_{xy}^2}{I_y} \qquad \text{(1a)}$$

$$I_{my} = \frac{I_x I_y - I_{xy}^2}{I_x} \qquad \text{(1b)}$$

9. Theory of Bending, Torsion and Buckling of Thin Walled Members of Open Cross-Section, by S. P. Timoshenko, Journal of the Franklin Institute, Vol. 239, No. 3, March 1945.

Eqns. (iii) and (ivb) can be written as:

$$\frac{1}{r_x} = -\frac{d^2\delta_y}{dz^2} = \frac{M_x}{EI_{mx}} \qquad \text{(v)}$$

and

$$\frac{1}{r_y} = -\frac{d^2\delta_x}{dz^2} = \frac{M_x(-I_{xy}/I_x)}{EI_{my}} \qquad \text{(vi)}$$

Eqn. (v) corresponds in form to the elementary differential equation for symmetrical bending. Therefore, by integrating it twice, corresponding expressions for the vertical deflections are obtained analogous to those for simple bending; it is only necessary to replace the moment of inertia in the elementary formulas for simple bending by the modified moment of inertia I_{mx} to obtain the desired vertical deflection. Thus, expressions similar to Eqn. (2a) can be formulated.

Eqn. (vi) indicates that the beam has a continuous curvature in the x-z plane when the real external loads are in the y-z plane. This curvature can be assumed to be caused by a fictitious bending moment M_y in the x-z plane. As Eqn. (vi) suggests, this fictitious bending moment should have the same diagram as the real bending moment M_x in the y-z plane, but reduced in magnitude throughout the beam by the factor $(-I_{xy}/I_x)$, or, $M_y = M_x(-I_{xy}/I_x)$. Further, since the fictitious bending moment diagram has the same shape as the real bending moment diagram, the fictitious bending moment can be assumed to have been produced by fictitious loads in the x-z plane, applied at the same locations along the beam as the real loads, but reduced in magnitude by the factor $(-I_{xy}/I_x)$. Hence, the horizontal deflections in the x-z plane caused by loads in the y-z plane can be computed by considering the beam loaded by the fictitious loads in the x-z plane and using the elementary deflection formulas for symmetrical bending in which the moment of inertia is replaced by the modified moment of inertia I_{my}. With such reasoning, expressions similar to Eqn. (2c) can be formulated.

(b) Loads in the x-z plane:

If a beam similar to that shown in Fig. 3, is loaded in the x-z plane, the only real external moment would be M_y about the y-axis. Because of oblique bending, however, the beam would also have a continuous curvature in the y-z plane. A discussion identical to that in connection with the previous case will show that:

$$\frac{1}{r_y} = -\frac{d^2\delta_x}{dz^2} = \frac{M_y}{EI_{my}} \qquad \text{(vii)}$$

and

$$\frac{1}{r_x} = -\frac{d^2\delta_y}{dz^2} = \frac{M_y(-I_{xy}/I_y)}{EI_{mx}} \qquad \text{(viii)}$$

and expressions similar to Eqns. (2f) and (2g) can be formulated.

(2) Stresses.

(a) Loads in the y-z plane:

Referring to the cross-section of the beam in Fig. 1a, the longitudinal stress, σ, at any point is obviously:

$$\sigma = \frac{E \cdot y}{r_x} + \frac{E \cdot x}{r_y} \tag{ix}$$

Substituting (v) and (vi) into (ix),

$$\sigma = \frac{M_x\ y}{I_{mx}} + \frac{M_x(-I_{xy}/I_x)\ x}{I_{my}} \tag{x}$$

which is of the same form as Eqn. (2e).

(b) Loads in the x-z plane (Fig. 3):

In this case,

$$\sigma = \frac{E\ x}{r_y} + \frac{E\ y}{r_x} \tag{xi}$$

Substituting (vii) and (viii) into (xi),

$$\sigma = \frac{M_y\ x}{I_{my}} + \frac{M_y(-I_{xy}/I_y)\ y}{I_{mx}} \tag{xii}$$

which is of the same form as Eqn. (2h).

(3) Braced Beams:

If a beam is braced laterally at some intermediate points, e.g. in the x-z plane as in Fig. 4a, there are real loads both in the y-z and the x-z planes. In the x-z plane, of course, the beam is statically indeterminate. Real loads in the y-z plane give rise to fictitious loads in the x-z plane, and vice versa. Thus, the beam in each plane is subjected simultaneously to real and to fictitious loads.

Eqns. (v), (vi), (vii) and (viii) for the deflections and Eqn. (x) and (xii) for the stresses, are linear functions of the external loads, and therefore can be superimposed. Hence, in computing deflections either in the y-z or in the x-z planes, both real and fictitious loads in the respective planes can be treated as one simultaneous system; the actual deflections of a beam in either plane, will then be due to the simultaneous system. This also indicates that the redundant reactions in the x-z plane (i.e. brace reactions) can be determined by the usual methods of analysis of continuous beams.

In computing stresses, on the other hand, Eqns. (x) and (xii) indicate that the actual bending stress at any point in the cross-section is equal to the superimposed bending stresses at the same point due to 1) the simultaneous system in the y-z plane and 2) the simultaneous system in the x-z plane.

From the above observations it may be concluded, therefore, that the beam in each plane may be treated independently; in each plane it should be loaded simultaneously by the real and the fictitious loads and the appropriate modified moment of inertia should be used, i.e. I_{mx} in the y-z plane and I_{my} in the x-z plane.

(4) Computations for the Maximum Stress in the Continuously Braced Beam:

Consider the beam in Fig. 5a to be restrained so that it can bend only in the y-z plane. This implies that the total bending moment due to the real and the fictitious loads in the x-z plane has to be nil throughout the beam span. (Note that in this case the real bending moment in the x-z plane is different from zero).

The fictitious bending moment in the x-z plane is shown in Fig. 6a. Since the resultant bending moment (fictitious plus real) should be zero throughout the span, the (real) bending moment due to the continuous bracing should, therefore, be exactly of the same shape and magnitude as the fictitious bending moment. Hence, the continuous bracing will exert two concentrated reactions at points "c" of 150 lbs each, as is shown dotted in Fig. 6a. These reactions, in turn, will give rise to two fictitious loads of 150 (I_{xy}/I_y) = 150 (2.6/1.41) = 276 lbs in the y-z plane, as shown in Fig. 6b.

We are now in a position to compute the bending stress at point "B" (Fig. 6). Since the loads in the x-z plane make no contribution to the bending stress at this point, only the loads in the y-z plane have to be considered.

The bending moment at midspan (Fig. 6b) is,

$$M_x = (522 - 276)\ 92 = 22630 \text{ lbs. in.} \qquad \text{(xiii)}$$

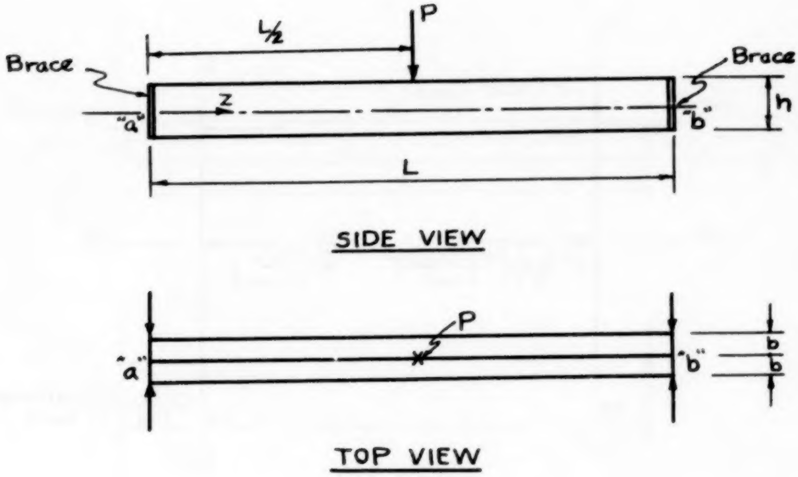
and therefore the stress at point "B" is:

$$\sigma = \frac{M_x\ (h/2)}{I_{mx}} = \frac{22630 \times 3.713}{4.28} = 19720 \text{ psi.} \qquad \text{(xiv)}$$

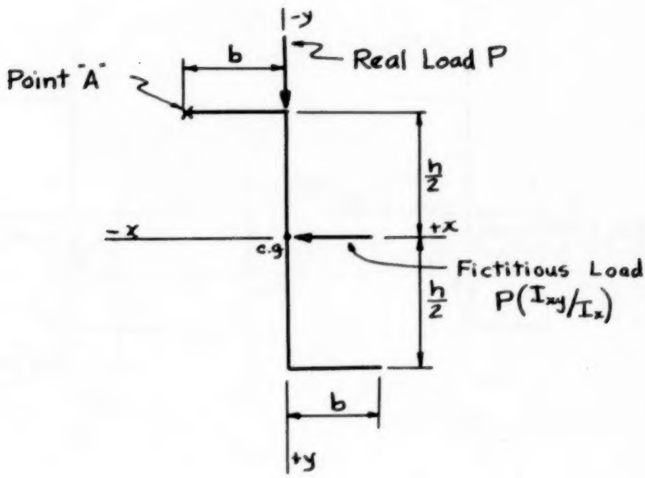
which is the same stress as obtained in Eqn. (8).

(5) Bracing in Beams Subjected to Concentrated Loads Only:

In the example just solved for the continuously braced beam, it was noticed that if a continuously braced beam is subjected only to concentrated loads, the reactions that the continuous bracing would exert will be concentrated forces located at exactly the same points where the external loads are applied. This suggests that if isolated lateral braces are placed at the locations of the vertically applied concentrated loads, the effect of the isolated braces on the beam will be same as that of continuous bracing. From this it can be concluded that in such a case, portions of the beam between braces will tend to bend only in the plane of the loads.


P
L/2
Brace
Brace
Z
"a"
"b"
h
L
SIDE VIEW
P
"a"
"b"
b
b
TOP VIEW

(a)


-y
Real Load P
Point "A"
b
h/2
-x
+x
c.g
Fictitious Load
P(Ixy/Ix)
h/2
b
+y

(b)

FIGURE 1

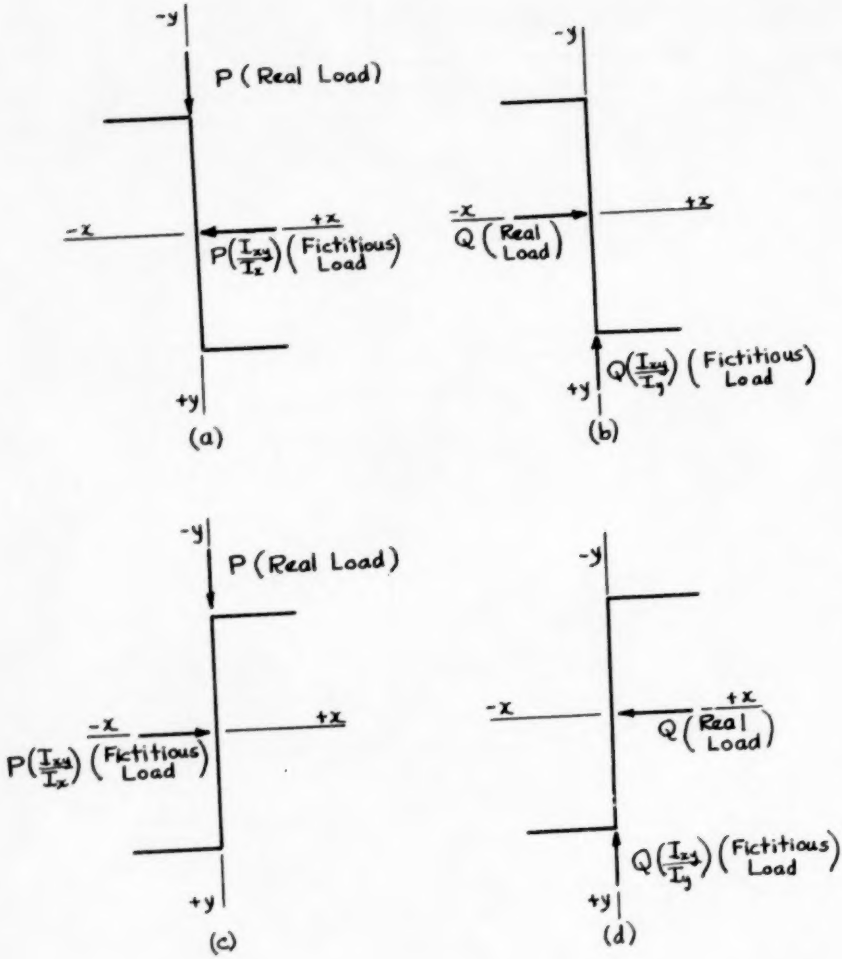
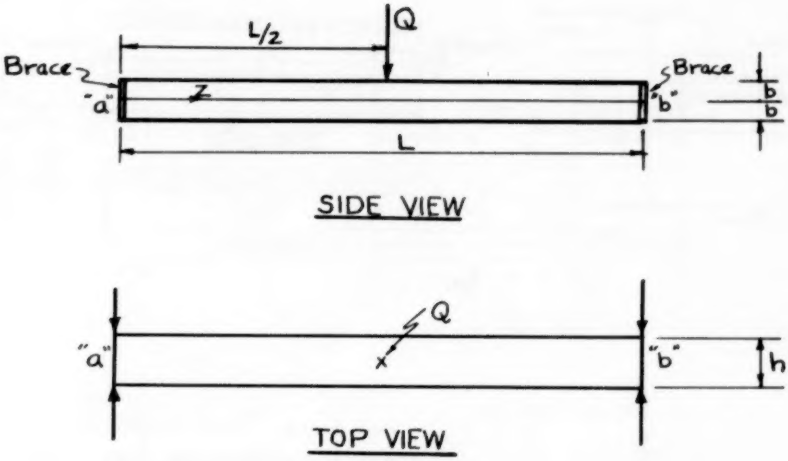
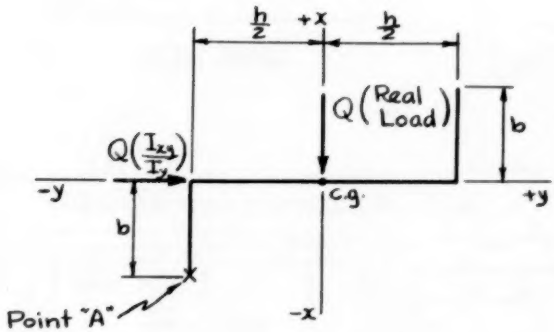

-y
P (Real Load)
-x
+x
P(Ixy/Ix) (Fictitious Load)
+y
(a)
-y
-x
+x
Q (Real Load)
Q(Ixy/Iy) (Fictitious Load)
+y
(b)
-y
P (Real Load)
-x
+x
P(Ixy/Ix) (Fictitious Load)
+y
(c)
-y
-x
+x
Q (Real Load)
Q(Ixy/Iy) (Fictitious Load)
+y
(d)

FIGURE 2

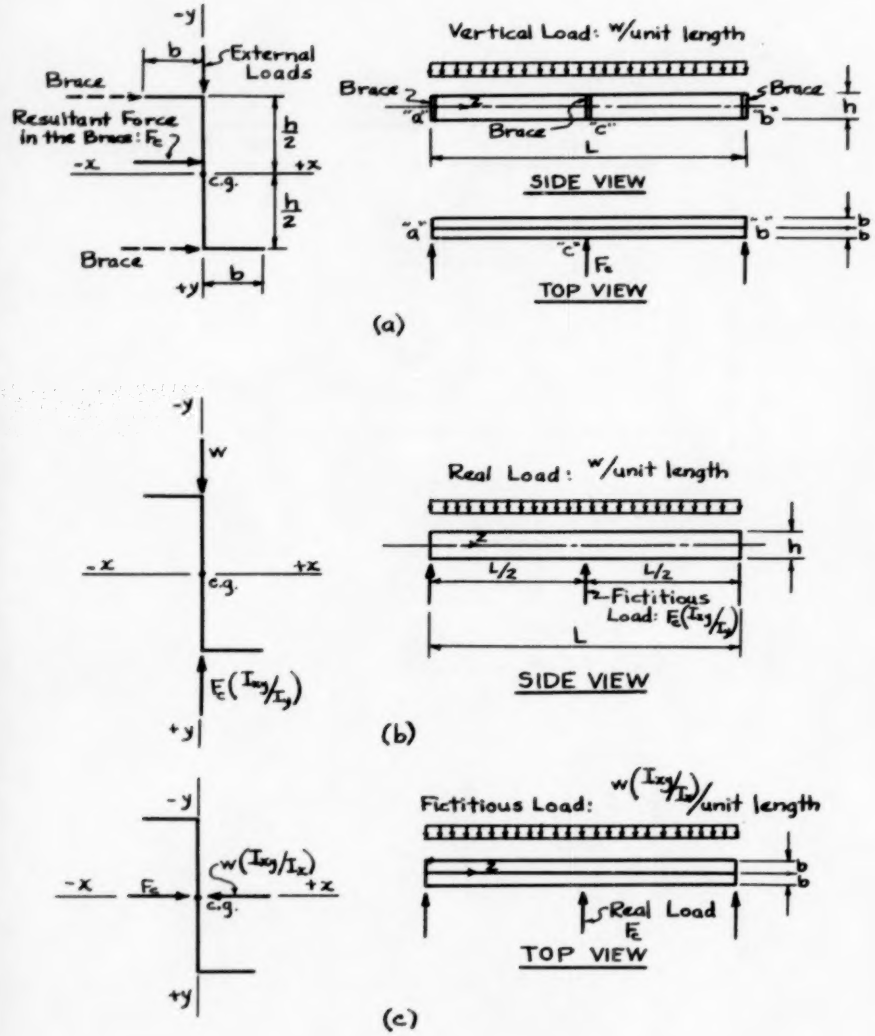

Q
L/2
Brace
Brace
"a"
"b"
z
b
b
L
SIDE VIEW
Q
"a"
"b"
x
h
TOP VIEW

(a)


h/2
+x
h/2
Q(Real Load)
b
Q(Ixy/Iy)
-y
c.g.
+y
b
Point "A"
-x

(b)

FIGURE 3



Brace
External Loads
Resultant Force in the Brace: F_c
-x
+x
c.g.
-y
+y
b
h/2
Vertical Load: w/unit length
Brace
"a"
"b"
"c"
h
L
SIDE VIEW
TOP VIEW
F_c
(a)
w
Real Load: w/unit length
L/2
Fictitious Load: $F_c(I_{xy}/I_y)$
(b)
Fictitious Load: $w(I_{xy}/I_x)$/unit length
$w(I_{xy}/I_x)$
Real Load F_c
(c)
z

FIGURE 4

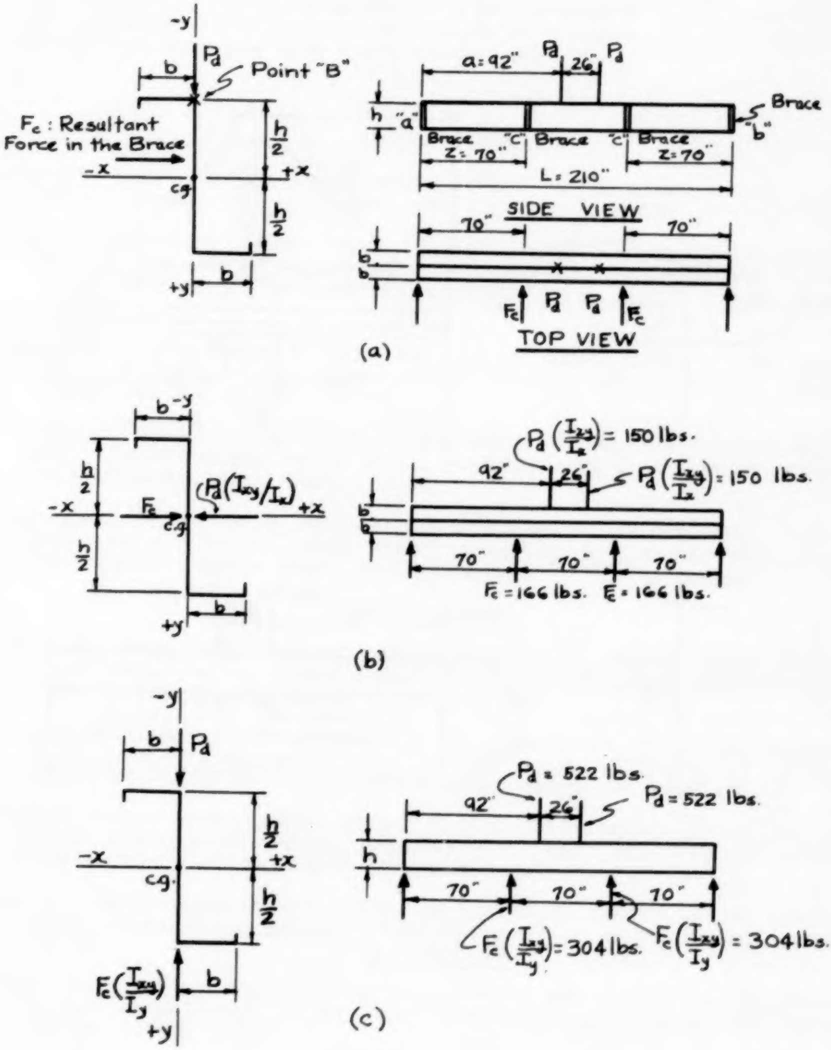

-y
P_d
b
Point "B"
F_c : Resultant Force in the Brace
-x
+x
cg
h/2
h/2
+y
b
a=92"
P_d
26"
P_d
h
"a"
Brace
"c"
Brace
"c"
Brace
Brace
"b"
Z= 70"
Z=70"
L= 210"
SIDE VIEW
70"
70"
b
b
F_c
P_d
P_d
F_c
TOP VIEW
(a)
b
-y
h/2
h/2
-x
+x
F_c
cg
$P_d(I_{xy}/I_x)$
+y
b
$P_d\left(\frac{I_{xy}}{I_x}\right)$= 150 lbs.
92"
26"
$P_d\left(\frac{I_{xy}}{I_x}\right)$= 150 lbs.
b
b
70"
70"
70"
F_c = 166 lbs.
F_c = 166 lbs.
(b)
-y
b
P_d
-x
+x
cg.
h/2
h/2
$F_c\left(\frac{I_{xy}}{I_y}\right)$
b
+y
P_d = 522 lbs.
P_d = 522 lbs.
92"
26"
h
70"
70"
70"
$F_c\left(\frac{I_{xy}}{I_y}\right)$= 304 lbs.
$F_c\left(\frac{I_{xy}}{I_y}\right)$ = 304 lbs.
(c)

FIGURE 5

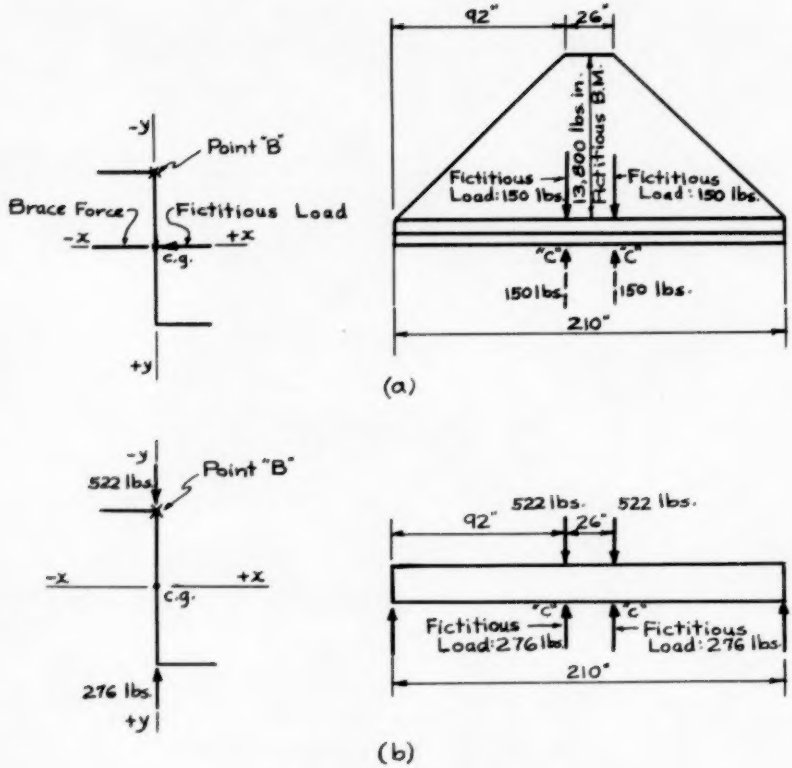

-y
Point "B"
Brace Force
Fictitious Load
-x
+x
c.g.
+y
92"
26"
13,800 lbs. in.
Fictitious B.M.
Fictitious Load: 150 lbs.
Fictitious Load: 150 lbs.
"C"
"C"
150 lbs.
150 lbs.
210"
(a)
-y
522 lbs.
Point "B"
-x
+x
c.g.
276 lbs.
+y
522 lbs.
522 lbs.
92"
26"
"C"
"C"
Fictitious Load: 276 lbs.
Fictitious Load: 276 lbs.
210"
(b)

FIGURE 6

PROCEEDINGS PAPERS

The technical papers published in the past year are presented below. Technical-division sponsorship is indicated by an abbreviation at the end of each Paper Number, the symbols referring to: Air Transport (AT), City Planning (CP), Construction (CO), Engineering Mechanics (EM), Highway (HW), Hydraulics (HY), Irrigation and Drainage (IR), Power (PO), Sanitary Engineering (SA), Soil Mechanics and Foundations (SM), Structural (ST), Surveying and Mapping (SU), and Waterways (WW) divisions. For titles and order coupons, refer to the appropriate issue of "Civil Engineering" or write for a cumulative price list.

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AUGUST: 466(HY), 467(HY), 468(ST), 469(ST), 470(ST), 471(SA), 472(SA), 473(SA), 474(SA), 475(SM), 476(SM), 477(SM), 478(SM)[c], 479(HY)[c], 480(ST)[c], 481(SA)[c], 482(HY), 483(HY).

SEPTEMBER: 484(ST), 485(ST), 486(ST), 487(CP)[c], 488(ST)[c], 489(HY), 490(HY), 491(HY)[c], 492(SA), 493(SA), 494(SA), 495(SA), 496(SA), 497(SA), 498(SA), 499(HW), 500(HW), 501(HW)[c], 502(WW), 503(WW), 504(WW)[c], 505(CO), 506(CO)[c], 507(CP), 508(CP), 509(CP), 510(CP), 511(CP).

OCTOBER: 512(SM), 513(SM), 514(SM), 515(SM), 516(SM), 517(PO), 518(SM)[c], 519(IR), 520(IR), 521(IR), 522(IR)[c], 523(AT)[c], 524(SU), 525(SU)[c], 526(EM), 527(EM), 528(EM), 529(EM), 530(EM)[c], 531(EM), 532(EM)[c], 533(PO).

NOVEMBER: 534(HY), 535(HY), 536(HY), 537(HY), 538(HY)[c], 539(ST), 540(ST), 541(ST), 542(ST), 543(ST), 544(ST), 545(SA), 546(SA), 547(SA), 548(SM), 549(SM), 550(SM), 551(SM), 552(SA), 553(SM)[c], 554(SA), 555(SA), 556(SA), 557(SA).

DECEMBER: 558(ST), 559(ST), 560(ST), 561(ST), 562(ST), 563(ST)[c], 564(HY), 565(HY), 566(HY), 567(HY), 568(HY)[c], 569(SM), 570(SM), 571(SM), 572(SM)[c], 573(SM)[c], 574(SU), 575(SU), 576(SU), 577(SU), 578(HY), 579(ST), 580(SU), 581(SU), 582(Index).

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JANUARY: 583(ST), 584(ST), 585(ST), 586(ST), 587(ST), 588(ST), 589(ST)[c], 590(SA), 591(SA), 592(SA), 593(SA), 594(SA), 595(SA)[c], 596(HW), 597(HW), 598(HW)[c], 599(CP), 600(CP), 601(CP), 602(CP), 603(CP), 604(EM), 605(EM), 606(EM)[c], 607(EM).

FEBRUARY: 608(WW), 609(WW), 610(WW), 611(WW), 612(WW), 613(WW), 614(WW), 615(WW), 616(WW), 617(IR), 618(IR), 619(IR), 620(IR), 621(IR)[c], 622(IR), 623(IR), 624(HY)[c], 625(HY), 626(HY), 627(HY), 628(HY), 629(HY), 630(HY), 631(HY), 632(CO), 633(CO).

MARCH: 634(PO), 635(PO), 636(PO), 637(PO), 638(PO), 639(PO), 640(PO), 641(PO)[c], 642(SA), 643(SA), 644(SA), 645(SA), 646(SA), 647(SA)[c], 648(ST), 649(ST), 650(ST), 651(ST), 652(ST), 653(ST), 654(ST)[c], 655(SA), 656(SM)[c], 657(SM)[c], 658(SM)[c].

APRIL: 659(ST), 660(ST), 661(ST)[c], 662(ST), 663(ST), 664(ST)[c], 665(HY)[c], 666(HY), 667(HY), 668(HY), 669(HY), 670(EM), 671(EM), 672(EM), 673(EM), 674(EM), 675(EM), 676(EM), 677(EM), 678(HY).

MAY: 679(ST), 680(ST), 681(ST), 682(ST)[c], 683(ST), 684(ST), 685(SA), 686(SA), 687(SA), 688(SA), 689(SA)[c], 690(EM), 691(EM), 692(EM), 693(EM), 694(EM), 695(EM), 696(PO), 697(PO), 698(SA), 699(PO)[c], 700(PO), 701(ST)[c].

JUNE: 702(HW), 703(HW), 704(HW)[c], 705(IR), 706(IR), 707(IR), 708(IR), 709(HY)[c], 710(CP), 711(CP), 712(CP), 713(CP)[c], 714(HY), 715(HY), 716(HY), 717(HY), 718(SM)[c], 719(HY)[c], 720(AT), 721(AT), 722(SU), 723(WW), 724(WW), 725(WW), 726(WW)[c], 727(WW), 728(IR), 729(IR), 730(SU)[c], 731(SU).

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AUGUST: 761(BD), 762(ST), 763(ST), 764(ST), 765(ST)[c], 766(CP), 767(CP), 768(CP), 769(CP), 770(CP), 771(EM), 772(EM), 773(SA), 774(EM), 775(EM), 776(EM)[c], 777(AT), 778(AT), 779(SA), 780(SA), 781(SA), 782(SA)[c], 783(HW), 784(HW), 785(CP), 786(ST).

c. Discussion of several papers, grouped by Divisions.

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